88147201

International Baccalaureate
Baccalauréat International
Bachillerato Internacional

MATHEMATICS
HIGHER LEVEL
PAPER 1

Candidate session number

Wednesday 12 November 2014 (afternoon)

2 hours

Examination code

8	8	1	4	–	7	2	0	1

INSTRUCTIONS TO CANDIDATES

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- You are not permitted access to any calculator for this paper.
- Section A: answer all questions in the boxes provided.
- Section B: answer all questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper and your cover sheet using the tag provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the ***Mathematics HL and Further Mathematics HL*** *formula booklet* is required for this paper.
- The maximum mark for this examination paper is *[120 marks]*.

14 pages

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

SECTION A

*Answer **all** questions in the boxes provided. Working may be continued below the lines if necessary.*

1. *[Maximum mark: 4]*

The function f is defined by $f(x) = \frac{1}{x}$, $x \neq 0$.

The graph of the function $y = g(x)$ is obtained by applying the following transformations to the graph of $y = f(x)$:

a translation by the vector $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$;

a translation by the vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

(a) Find an expression for $g(x)$. *[2]*

(b) State the equations of the asymptotes of the graph of g. *[2]*

2. *[Maximum mark: 6]*

The quadratic equation $2x^2 - 8x + 1 = 0$ has roots α and β.

(a) Without solving the equation, find the value of

(i) $\alpha + \beta$;

(ii) $\alpha\beta$. *[2]*

Another quadratic equation $x^2 + px + q = 0$, $p, q \in \mathbb{Z}$ has roots $\dfrac{2}{\alpha}$ and $\dfrac{2}{\beta}$.

(b) Find the value of p and the value of q. *[4]*

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3. *[Maximum mark: 5]*

A point P, relative to an origin O, has position vector $\overrightarrow{OP} = \begin{pmatrix} 1+s \\ 3+2s \\ 1-s \end{pmatrix}$, $s \in \mathbb{R}$.

Find the minimum length of $\overrightarrow{OP}$.

4. *[Maximum mark: 7]*

Events A and B are such that $\mathrm{P}(A) = 0.2$ and $\mathrm{P}(B) = 0.5$.

(a) Determine the value of $\mathrm{P}(A \cup B)$ when

(i) A and B are mutually exclusive;

(ii) A and B are independent. *[4]*

(b) Determine the range of possible values of $\mathrm{P}(A|B)$. *[3]*

5. *[Maximum mark: 6]*

A tranquilizer is injected into a muscle from which it enters the bloodstream. The concentration C in $\text{mg}\,\text{l}^{-1}$, of tranquilizer in the bloodstream can be modelled by the function $C(t) = \dfrac{2t}{3+t^2}$, $t \geq 0$ where t is the number of minutes after the injection.

Find the maximum concentration of tranquilizer in the bloodstream.

6. *[Maximum mark: 6]*

By using the substitution $u = 1 + \sqrt{x}$, find $\int \frac{\sqrt{x}}{1+\sqrt{x}}\,\mathrm{d}x$.

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7. *[Maximum mark: 6]*

Consider two functions f and g and their derivatives f' and g'. The following table shows the values for the two functions and their derivatives at $x = 1$, 2 and 3.

x	1	2	3
$f(x)$	3	1	1
$f'(x)$	1	4	2
$g(x)$	2	1	4
$g'(x)$	4	2	3

Given that $p(x) = f(x)\,g(x)$ and $h(x) = g \circ f(x)$, find

(a) $p'(3)$; *[2]*

(b) $h'(2)$. *[4]*

8. *[Maximum mark: 7]*

Use mathematical induction to prove that $(2n)! \geq 2^n (n!)^2$, $n \in \mathbb{Z}^+$.

9. *[Maximum mark: 6]*

A continuous random variable T has probability density function f defined by

$$f(t) = \begin{cases} |2-t|, & 1 \le t \le 3 \\ 0, & \text{otherwise}. \end{cases}$$

(a) Sketch the graph of $y = f(t)$. *[2]*

(b) Find the interquartile range of T. *[4]*

10. *[Maximum mark: 7]*

A set of positive integers $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ is used to form a pack of nine cards. Each card displays one positive integer without repetition from this set. Grace wishes to select four cards at random from this pack of nine cards.

(a) Find the number of selections Grace could make if the largest integer drawn among the four cards is either a 5, a 6 or a 7. *[3]*

(b) Find the number of selections Grace could make if at least two of the four integers drawn are even. *[4]*

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SECTION B

*Answer **all** questions in the answer booklet provided. Please start each question on a new page.*

11. *[Maximum mark: 23]*

The function f is defined as $f(x) = e^{3x+1}$, $x \in \mathbb{R}$.

(a) (i) Find $f^{-1}(x)$.

(ii) State the domain of f^{-1}. *[4]*

The function g is defined as $g(x) = \ln x$, $x \in \mathbb{R}^+$.
The graph of $y = g(x)$ and the graph of $y = f^{-1}(x)$ intersect at the point P.

(b) Find the coordinates of P. *[5]*

The graph of $y = g(x)$ intersects the x-axis at the point Q.

(c) Show that the equation of the tangent T to the graph of $y = g(x)$ at the point Q is $y = x - 1$. *[3]*

A region R is bounded by the graphs of $y = g(x)$, the tangent T and the line $x = e$.

(d) Find the area of the region R. *[5]*

(e) (i) Show that $g(x) \le x - 1$, $x \in \mathbb{R}^+$.

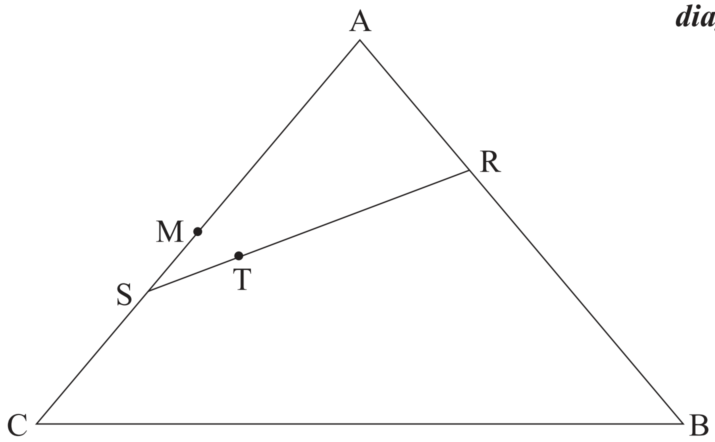
(ii) By replacing x with $\frac{1}{x}$ in part (e)(i), show that $\frac{x-1}{x} \le g(x)$, $x \in \mathbb{R}^+$. *[6]*

*Do **NOT** write solutions on this page.*

12. *[Maximum mark: 14]*

The position vectors of the points A, B and C are $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ respectively, relative to an origin O. The following diagram shows the triangle ABC and points M, R, S and T.

diagram not to scale



M is the midpoint of [AC].

R is a point on [AB] such that $\overrightarrow{AR} = \frac{1}{3}\overrightarrow{AB}$.

S is a point on [AC] such that $\overrightarrow{AS} = \frac{2}{3}\overrightarrow{AC}$.

T is a point on [RS] such that $\overrightarrow{RT} = \frac{2}{3}\overrightarrow{RS}$.

(a) (i) Express $\overrightarrow{AM}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{c}$.

(ii) Hence show that $\overrightarrow{BM} = \frac{1}{2}\boldsymbol{a} - \boldsymbol{b} + \frac{1}{2}\boldsymbol{c}$. *[4]*

(b) (i) Express $\overrightarrow{RA}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

(ii) Show that $\overrightarrow{RT} = -\frac{2}{9}\boldsymbol{a} - \frac{2}{9}\boldsymbol{b} + \frac{4}{9}\boldsymbol{c}$. *[5]*

(c) Prove that T lies on [BM]. *[5]*

*Do **NOT** write solutions on this page.*

13. *[Maximum mark: 23]*

(a) (i) Show that $(1+\mathrm{i}\tan\theta)^n+(1-\mathrm{i}\tan\theta)^n=\dfrac{2\cos n\theta}{\cos^n\theta}$, $\cos\theta\neq 0$.

(ii) Hence verify that $\mathrm{i}\tan\dfrac{3\pi}{8}$ is a root of the equation $(1+z)^4+(1-z)^4=0$, $z\in\mathbb{C}$.

(iii) State another root of the equation $(1+z)^4+(1-z)^4=0$, $z\in\mathbb{C}$. *[10]*

(b) (i) Use the double angle identity $\tan 2\theta=\dfrac{2\tan\theta}{1-\tan^2\theta}$ to show that $\tan\dfrac{\pi}{8}=\sqrt{2}-1$.

(ii) Show that $\cos 4x=8\cos^4 x-8\cos^2 x+1$.

(iii) Hence find the value of $\displaystyle\int_0^{\frac{\pi}{8}}\frac{2\cos 4x}{\cos^2 x}\,\mathrm{d}x$. *[13]*

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will not be marked.

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